The Calculus of Variations

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Why Do We Care?

What does an esoteric mathematical construct, such as the calculus of variations, have to do with audio? There is one very simple reason why this subject and countless others demand our attention, rather than our quick dismissal:

You never know where your next big inspiration will come from.

Everything is interrelated. The deeper you dig the more you realize that the answers in one subject bear an uncanny resemblance to those in a subject seemingly far removed. Take a tour through the Mandelbrot set sometime and you'll get an intuitive sense of what I am talking about as you stumble across across many “self-similar” structures. Structures that are never quite the same, but you know without a doubt that you have seen them *somewhere* before.

What Are The Applications?

- Classical mechanics equations of motion
- Optimal control (minimize cost function)
- Robotic inverse kinematic controllers
- Fermat's principle for optics
- Quantum mechanics (sum over path)
- Motion of charged particle in field

This is a partial list – there are many more areas where this useful mathematical tool has been applied. If you are in a mathematical mood, then please work through the following derivation. You never know when this knowledge might come in handy, regardless of whether your passion is archeology or amplifiers!

What Is The Question?

We are often interested in *extrema*: What is the fastest route to deliver these newspapers? What gives the lowest total cost for this product? Which stein holds the most beer? The calculus of variations puts this sort of question into the language of mathematics, thus allowing application of the tools built up over the last few millennia. Okay, now for the formal bit:

Determine $y(x)$ such that $J=\int_{x_1}^{x_2} f\{y(x), y'(x); x\}$ is an extremum (minimum or maximum).

Let $y(\alpha, x)=y(0,x)+\alpha\,\eta(x)$ where $y(0,\alpha)=y(x)$ is the extremum, all $\alpha\neq 0$ are paths that depart from the extremum, and $\eta(x)$ is a function of x that has continuous first derivative and vanishes for x_1 and x_2 .

Please refer to Fig. 1 for an illustration of this construction.

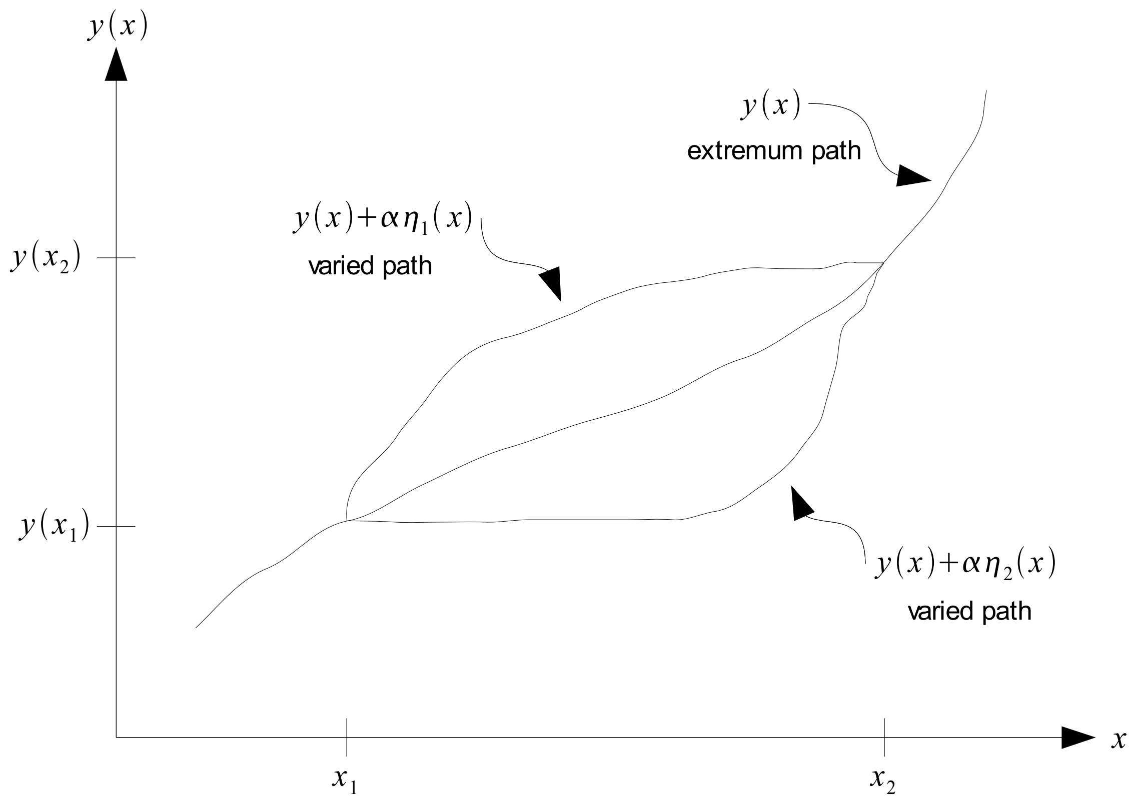

Figure 1 – illustration of extremum path $y(x)$ and varied paths $y(x)+\alpha\,\eta(x)$

What Is The Answer?

In order for $J=\int_{x_1}^{x_2} f\{y(x), y'(x); x\}$ to have an extremum, it is necessary that $\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0}=0$.

$$\frac{\partial J}{\partial \alpha}=\frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\{y(x), y'(x); x\}dx$$ the limits of integration are fixed so

$$\frac{\partial J}{\partial \alpha}=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right)dx$$ the partial derivatives are given by

$$\frac{\partial y}{\partial \alpha}=\frac{\partial}{\partial \alpha}(y(x)+\alpha\eta(x))=\eta(x) \quad ; \quad \frac{\partial y'}{\partial \alpha}=\frac{\partial}{\partial \alpha}\left(\frac{d}{dx}(y(x)+\alpha\eta(x))\right)=\eta'(x)$$

$$\frac{\partial J}{\partial \alpha}=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)\right)dx$$ using integration by parts $\int u\, dv=uv-\int v\, du$ on the second term

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'(x)\,dx=\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\frac{d\eta(x)}{dx}dx=\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}d\,\eta(x)$$ now let $u=\frac{\partial f}{\partial y'}$ and $dv=d\,\eta(x)$ then

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'(x)\,dx=\left.\frac{\partial f}{\partial y'}\eta(x)\right|_{x_1}^{x_2}-\int_{x_1}^{x_2}\eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$ note that $\eta(x_1)=\eta(x_2)=0$ so

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\eta'(x)\,dx=-\int_{x_1}^{x_2}\eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$ now using this result in the expression for $\frac{\partial J}{\partial \alpha}$ yields

$$\frac{\partial J}{\partial \alpha}=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x)-\eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)dx=\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)\,dx$$

The next step makes use of the Fundamental Lemma of the Calculus of Variations:

If $\int_a^b m(x)h(x)\,dx=0$ for all $h(x)\neq 0$ with continuous 2nd partial derivatives, then $m(x)=0$ on $[a,b]$.

Since $\eta(x)$ is an arbitrary function subject to $\eta(x_1)=\eta(x_2)=0$ (i.e. $\eta(x)\neq 0$) and $\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0}=0$ in order for an extremum to exist, by the fundamental lemma of the calculus of variations it is necessary that

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0$$ this result is known as the Euler-Lagrange Equation.

J has an extremum if the Euler-Lagrange equation is satisfied (i.e. it is a necessary, but not sufficient, condition).